

EQUIVALENT NOTIONS OF NORMAL QUANTUM SUBGROUPS, COMPACT QUANTUM GROUPS WITH PROPERTIES F AND FD , AND OTHER APPLICATIONS

SHUZHOU WANG

ABSTRACT. It is shown that the notion of normal quantum subgroup introduced in purely algebraic context by Parshall and Wang when applied to compact quantum groups is equivalent to the notion of normal quantum subgroup introduced by the author. As applications, we obtain a quantum analog of the third fundamental isomorphism theorem for groups, which is used along with the above equivalence theorem to obtain general results on quantum groups with property F and quantum groups with property FD . Other results on normal quantum subgroups for tensor products, free products and crossed products are also proved.

1. INTRODUCTION

The notion of normal quantum subgroup is an important and subtle concept in the theory of quantum groups. In purely algebraic context of Hopf algebras, B. Parshall and J. Wang [11] defined a notion of normal quantum subgroup using left and right coadjoint actions of the Hopf algebra on itself, which was further studied by other authors such as Schneider [14], Takeuchi [17], and Andruskiewitsch and Devoto [1]. Parshall and Wang noted that left normal quantum groups need not be right normal in general, and if an exact sequence exists, it may not be unique. These difficulties are peculiar phenomena of general Hopf algebras in purely algebraic context and it distinguishes Hopf algebras from groups. Other complications related to the notion of normal quantum groups in purely algebraic context are included in [14]. In C^* -algebraic context, the author introduced [20] a notion of normal quantum subgroup of compact quantum groups using analytical properties of representation theory of compact quantum groups. It was not known whether these two notions of normality are equivalent

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when they are applied to the canonical dense Hopf $*$ -algebras of quantum representative functions of compact quantum groups. In [25], the author's notion of normal quantum groups was used in an essential way to define the notion of simple compact quantum groups. It was also announced in [25] without proof that the above two notions of normality are equivalent (see remark (b) after Lemma 4.4 in [25]) for compact quantum groups. As consequences, left normal and right normal defined in algebraic context by Pashall and Wang are also equivalent for Hopf algebras associated with compact quantum groups, and their normal quantum subgroups always give rise to a unique exact sequence. That is, the complications mentioned above in purely algebraic setting do not present themselves in the world of compact quantum groups. The associated property might be useful for formulating an appropriate notion of quantum groups in purely algebraic setting, which is still an open problem. Other facts announced in [25] without proofs include general structural results on quantum subgroups and quotient quantum groups of a compact quantum group with property F (resp. property FD), where, roughly speaking, a compact quantum group G is said to have property F if its quantum function algebra A_G has the same the property with respect to quotients by normal quantum subgroups as the function algebra of a compact group, and it is said to have property FD if its quantum function algebra has the same property with respect to quotients by normal quantum subgroups as the quantum function algebra of the dual of a discrete group. See Definition 4.2 below for precise definitions of these concepts and notation used above. Compact quantum groups with property F include all quantum groups obtained from compact Lie groups by deformation method, such as compact real form of the Drinfeld-Jimbo quantum groups and Rieffel's deformation, as well as most of the universal quantum groups constructed by the author except the universal unitary quantum groups $A_u(Q)$ (also called the free unitary quantum groups), cf. [25].

The purposes of this paper are to give complete proof of the equivalence of the two notions of normality mentioned above and give the following applications of this Equivalence Theorem on the structure of compact quantum groups.

(1) We establish a complete quantum analog of the third fundamental isomorphism theorem. This is the only one among the three fundamental isomorphism theorems that has a complete quantum analog without added conditions or restrictions. On the contrary, a surjection of compact quantum

groups (i.e. inclusion of Woronowicz C^* -algebras) does not always give rise to a quantum analog of the first fundamental isomorphism theorem, except in the special case where an exact sequence can be constructed, cf. [11, 14, 17, 1] for this and other subtleties. Taking the example of the group C^* -algebra $A_G := C^*(F_2)$ of the free group F_2 on two generators, a Woronowicz C^* -subalgebra of A_G does not give rise to an exact sequence unless it is the group C^* -algebra of a normal subgroup of F_2 . In addition, it is not clear at the moment how a quantum analog of the second fundamental isomorphism theorem can be formulated.

(2) Using the Equivalence Theorem and the quantum analog of the third fundamental isomorphism theorem, we show that quantum subgroups and quotient quantum groups of a compact quantum group with property F also have property F , quantum subgroups of a compact quantum group with property FD also have property FD , and quotient quantum groups of a compact quantum group with property FD also have property FD provided G also has the pull back property. The pull back property is the quantum group version of the group situation in which every subgroup of G/N is of the form H/N for some subgroup H of G containing N . We believe all compact quantum groups have pull back property.

(3) We prove results on normal quantum subgroups for tensor products, free products and crossed products. Note that the free product construction has no place in the classical world of compact groups, it is a total quantum phenomenon.

An outline of the paper is as follows. In section 2, we recall the algebraic notion of normal quantum subgroups in [11] and the analytical notion of normal quantum subgroups in [20] respectively. In section 3, the equivalence of these two notions of normality is proven. The proof relies on reduction of quantum subgroups from C^* -algebraic setting to algebraic setting and the existence of splitting morphism for surjective morphisms between comodules over Hopf algebras associated with compact quantum groups, thanks to Woronowicz's fundamental work [26]. The reduction and splitting morphism are used in the proof of the main Lemma 3.3 for the theorem on equivalence. In sections 4, as applications of the equivalence theorem, we prove a quantum analog of the third fundamental isomorphism theorem, and show that quantum subgroups and quotient quantum groups of a compact quantum group with property F also have property F , and quantum subgroups and quotient quantum groups of a compact quantum group with property FD

also have property FD provided G also has pull back property. In section 5, as further applications, several properties of normal quantum subgroups for free products, tensor products and crossed products are given.

Besides the general abstract theory on compact quantum groups developed by Woronowicz and general constructions of particular classes of compact quantum groups, there seem to be no general results on the *structure* of *infinite* compact quantum groups in the literature with the possible exception of [6], a situation contrary to finite quantum groups for which there is much literature on their structure and classification. The results in sections 4 and 5 are a modest attempt at developing theory on structure of infinite compact quantum groups. It is expected that such results will be useful in the program of classification of simple compact quantum groups and further study of structure of compact quantum groups.

Convention: We use the notation and terminology in [20]. For a compact quantum group G , A_G denotes the underlying Woronowicz C^* -algebra and \mathcal{A}_G the associated canonical dense Hopf $*$ -algebra of quantum representative functions on G . Sometimes it is convenient to abuse the notation by calling A_G a compact quantum group, referring to G . As was pointed out on p.533 of [22], morphisms between quantum groups are meaningful only for *full* Woronowicz C^* -algebras A_G (i.e. restriction of the norm $\|\cdot\|$ of A_G to the $*$ -algebra \mathcal{A}_G is the maximum of all possible C^* -norm on \mathcal{A}_G), although one can define morphisms between arbitrary Woronowicz C^* -algebras (cf. 2.3 in [20]). Unless otherwise explicitly stated, we assume that all Woronowicz C^* -algebras considered in this note to be full. We also use standard notation in Hopf algebras, including Sweedler's summation convention [15, 10], and Δ , S , ε for coproduct, antipode and counit, respectively. Relevant basic information on compact quantum groups and Hopf algebras can also be found in [9].

2. TWO NOTIONS OF NORMAL QUANTUM SUBGROUPS AND THEIR EQUIVALENCE

We recall the two notions of normal quantum subgroups as defined by the author in [20] and by Parshall and Wang in [11].

A quantum subgroup of a compact quantum group G in the sense of [20] (see Definition 2.13 therein) is a pair (N, π) , where A_N is a Woronowicz C^* -algebra and $\pi : A_G \longrightarrow A_N$ is a surjection of C^* -algebras that satisfies

$$\Delta_N \pi = (\pi \otimes \pi) \Delta_G,$$

where Δ_G and Δ_N are the coproducts of A_G and A_N respectively. Denote by $\hat{\pi} : A_G \rightarrow A_N$ the corresponding morphism from the algebra A_G of quantum representative functions on G to that A_N on N . The quantum group (N, π) should be more precisely called a closed quantum subgroup, but we will omit the word *closed* since we do not consider non-closed quantum subgroups.

Definition 2.1. (cf. Definition 2.13 in [20]) *A quantum subgroup N of G is called **normal** if for every irreducible representation u^λ of G , the multiplicity of the trivial representation of N in $(id \otimes \pi)u^\lambda$ is equal to either zero or the dimension of u^λ .*

Let h_N be the Haar measure on N . Then it is clear that N is normal if and only if for every irreducible representation u^λ of G , either $h_N \pi(u^\lambda) = I_{d_\lambda}$ or $h_N \pi(u^\lambda) = 0$, where d_λ is the dimension of u^λ and I_{d_λ} is the $d_\lambda \times d_\lambda$ identity matrix.

To recall the definition of normal quantum subgroup in Parshall and Wang [11], we restrict our attention to Hopf algebras of the form A_G where G is a compact quantum group, though the definition is valid for more general Hopf algebras. An **algebraic quantum subgroup** of G in the sense of [11] is a pair (N, η) where A_N is a Hopf algebra (associated with a compact quantum under our restriction) and $\eta : A_G \rightarrow A_N$ is a surjection of Hopf algebras that satisfies

$$\Delta_N \eta = (\eta \otimes \eta) \Delta_G.$$

It is clear that if (N, π) is a quantum subgroup of G in the sense of [20], then $(N, \hat{\pi})$ is an algebraic quantum subgroup of G . However it must be cautioned that if (N, η) is an algebraic quantum subgroup of G and both G and N are compact, unless η preserves the $*$ -structures of A_G and A_N , there is *no* morphism of compact quantum groups $\pi : A_G \rightarrow A_N$ with $\hat{\pi} = \eta$.

The precise correspondence between analytical normal quantum subgroups and algebraic normal quantum subgroups of G is given by the following theorem (see 4.3.(2) in [25]), which is the first step that reduces the C^* -setting to algebraic setting for the proof of our equivalence theorem on normality:

Theorem 2.2. *The map $f(\mathcal{I}) = \overline{\mathcal{I}}$ is a bijection from the set of Hopf $*$ -ideals $\{\mathcal{I}\}$ of A_G onto the set of Woronowicz C^* -ideals $\{I\}$ of A_G with full quotient Woronowicz C^* -algebra A_G/I . The inverse g of f is given by $g(I) = I \cap A_G$.*

Remarks: A detailed proof of the above theorem is give in [25]. We note that its proof is a nice interplay between the algebraic and analytical properties of

compact quantum groups. We also note that many concrete constructions in the analytical C^* -algebraic context also have purely algebraic formulation, such as the quantum permutation groups in [23] and their algebraic counterpart in Bichon [2, 3]. The theory of compact quantum groups is a rich ground where algebraic aspects and analytical aspects pleasantly interplay with each other.

Thanks to Theorem 2.2, we now focus on the algebraic object \mathcal{A}_G . Let $a \in \mathcal{A}_G$. The left and right adjoint actions are defined by

$$ad_l(a) := \sum a_{(2)} \otimes a_{(1)} S(a_{(3)}), \quad ad_r(a) := \sum a_{(2)} \otimes S(a_{(1)}) a_{(3)},$$

where S is the antipode of the Hopf algebra \mathcal{A}_G and Sweedler's notation [15] is used:

$$(\Delta \otimes id)\Delta(a) = (id \otimes \Delta)\Delta(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}.$$

Definition 2.3. (cf. Definition 1.5 in [11]) *A algebraic quantum subgroup (N, η) of G is called **a-normal** if $\ker(\eta)$ is a normal Hopf ideal of \mathcal{A}_G , that is,*

$$ad_l(a) \in \ker(\eta) \otimes \mathcal{A}_G, \quad \text{and} \quad ad_r(a) \in \ker(\eta) \otimes \mathcal{A}_G$$

for all $a \in \ker(\eta)$.

For the time being we use the term **a-normal** for the situation considered by Parshall and Wang [11]. In Schneider [14], a morphism such as η used in the definition above is also called **conormal** morphism.

Our *first goal* in this paper is to prove the following equivalence theorem.

Theorem 2.4. *Let (N, π) be a quantum subgroup of a compact quantum group G . Then (N, π) is normal if and only if $(N, \hat{\pi})$ is a-normal.*

Note that the “if” part in Theorem 2.4 is immediate. Namely, let (N, η) be an a-normal quantum subgroup of G , where G and N are compact quantum groups, then there is a morphism $\pi : A_G \rightarrow A_N$ of compact quantum groups such that $\hat{\pi} = \eta$, because a $*$ -morphism η at the algebraic level always extends to a morphism of full Woronowicz C^* -algebras by the definition of the universal norm.

3. PROOF OF THEOREM 2.4

For convenience of the reader, we recall the notations to be used below. Define

$$A_{G/N} = \{a \in A_G \mid (id \otimes \pi)\Delta(a) = a \otimes 1_N\},$$

$$A_{N \setminus G} = \{a \in A_G \mid (\pi \otimes id)\Delta(a) = 1_N \otimes a\},$$

where Δ is the coproduct on A_G , id is the identity map on A_G , and 1_N is the unit of the algebra A_N , which will simply be denoted by 1 when the context is clear. Similarly, we define

$$\mathcal{A}_{G/N} = \mathcal{A}_G \cap A_{G/N}, \quad \text{and} \quad \mathcal{A}_{N \setminus G} = \mathcal{A}_G \cap A_{N \setminus G}.$$

Note that G/N and $N \setminus G$ should be denoted more precisely by $G/(N, \pi)$ and $(N, \pi) \setminus G$ respectively if there is a possible confusion. Let h_N be the Haar measure on N . Let

$$E_{G/N} = (id \otimes h_N \pi)\Delta, \quad E_{N \setminus G} = (h_N \pi \otimes id)\Delta.$$

Then $E_{G/N}$ and $E_{N \setminus G}$ are projections of norm one (completely positive and completely bounded conditional expectations) from A_G onto $A_{N \setminus G}$ and $A_{G/N}$ respectively (cf. [12] as well as Proposition 2.3 and Section 6 of [24]), and

$$\mathcal{A}_{G/N} = E_{G/N}(\mathcal{A}_G), \quad \text{and} \quad \mathcal{A}_{N \setminus G} = E_{N \setminus G}(\mathcal{A}_G).$$

The proposition below follows immediately from the above considerations.

Proposition 3.1. *The $*$ -subalgebras $\mathcal{A}_{N \setminus G}$ and $\mathcal{A}_{G/N}$ are dense in $A_{G/N}$ and $A_{N \setminus G}$ respectively under the norm of A_G .*

From Proposition 3.1, we have following slight reformulation of Proposition 2.1 in [25]:

Proposition 3.2. *Let N be a quantum subgroup of a compact quantum group G . Then the following conditions are equivalent:*

- (1) $A_{N \setminus G}$ is a Woronowicz C^* -subalgebra of A_G .
- (1)' $\mathcal{A}_{N \setminus G}$ is a Hopf $*$ -subalgebra of \mathcal{A}_G .
- (2) $A_{G/N}$ is a Woronowicz C^* -subalgebra of A_G .
- (2)' $\mathcal{A}_{G/N}$ is a Hopf $*$ -subalgebra of \mathcal{A}_G .
- (3) $A_{G/N} = A_{N \setminus G}$.
- (3)' $\mathcal{A}_{G/N} = \mathcal{A}_{N \setminus G}$.
- (4) N is normal.

Because of Theorem 2.2, Proposition 3.1 and Proposition 3.2, we may (and will) work exclusively with the dense Hopf $*$ -algebras of Woronowicz C^* -algebras from now on unless otherwise specified. As remarked after the proof of Proposition 2.1 in [25], the counit of $\mathcal{A}_{G/N}$ is equal to the restriction morphism $\pi|_{\mathcal{A}_{G/N}}$.

As usual, if \mathcal{H} is a Hopf algebra, \mathcal{H}^+ denotes the augmentation ideal (i.e. \mathcal{H}^+ is kernel of the counit ε of \mathcal{H}). Assume N is a normal quantum subgroup of a compact quantum group G . Then we have a Hopf $*$ -algebra $\mathcal{A}_{G/N}$ and its augmentation ideal $\mathcal{A}_{G/N}^+$.

Lemma 3.3 below is a key ingredient in the proof of Theorem 2.4. In the case of an ordinary compact group G , its geometric meaning is the trivial fact that a normal subgroup N of G is the inverse image of the identity element in G/N under the quotient map. However, in the case of quantum groups using the Hopf algebra language, it is a rather technical to prove and no simpler proof seems to be available. See Example 1.2 in Schneider [14] for other complications involved concerning the notion of normality for arbitrary Hopf algebras.

Lemma 3.3. (Reconstruction of N from identity in G/N)

Let (N, π) be a normal quantum subgroup of a compact quantum group G . Let $\hat{\pi} = \pi|_{\mathcal{A}_G}$ be the associated morphism from \mathcal{A}_G to \mathcal{A}_N . Then,

$$\ker(\hat{\pi}) = \mathcal{A}_{G/N}^+ \mathcal{A}_G = \mathcal{A}_G \mathcal{A}_{G/N}^+ = \mathcal{A}_G \mathcal{A}_{G/N}^+ \mathcal{A}_G.$$

Proof. A sketch of the proof is given in [25]. The method of proof is an adaption of 16.0.2 in Sweedler [15] and (4.21) in Childs [5] where Hopf algebras are assumed to be *finite* dimensional, whereas our Hopf algebras here are *infinite dimensional* in general. Because of other important related problems besides the ones we deal with herein, we provide full details below on how to adapt the existing method to infinite dimensional case.

It suffices to prove $\ker(\hat{\pi}) = \mathcal{A}_{G/N}^+ \mathcal{A}_G$, as we will have equality $\ker(\hat{\pi}) = \mathcal{A}_G \mathcal{A}_{G/N}^+$ by the same method, and these will imply that

$$\mathcal{A}_G \mathcal{A}_{G/N}^+ \mathcal{A}_G = \mathcal{A}_{G/N}^+ \mathcal{A}_G \mathcal{A}_G = \mathcal{A}_{G/N}^+ \mathcal{A}_G = \ker(\hat{\pi}) = \mathcal{A}_G \mathcal{A}_{G/N}^+.$$

Consider the right \mathcal{A}_N -comodule structures α and β on \mathcal{A}_G and \mathcal{A}_N defined respectively by

$$\alpha = (id \otimes \hat{\pi}) \Delta_G : \mathcal{A}_G \rightarrow \mathcal{A}_G \otimes \mathcal{A}_N,$$

$$\beta = \Delta_N : \mathcal{A}_N \rightarrow \mathcal{A}_N \otimes \mathcal{A}_N,$$

where Δ_G and Δ_N are respectively the coproducts of the Hopf algebras \mathcal{A}_G and \mathcal{A}_N . Since $\hat{\pi}$ is compatible with the coproducts, one verifies that $(\hat{\pi} \otimes id)\alpha = \beta\hat{\pi}$. That is, the surjection $\hat{\pi}$ is a morphism of \mathcal{A}_N -comodules from \mathcal{A}_G to \mathcal{A}_N . The Hopf algebra \mathcal{A}_N is cosemisimple by the fundamental work of Woronowicz [26] (see remarks in 2.2 of [20] which assures work in [26]

is valid for all compact quantum groups without separability assumption on the underlying C^* -algebra \mathcal{A}_G because of Van Daele's theorem [18] on the Haar measure based on [26, 27]). Therefore it follows from Theorem 3.1.5 of [7] that every \mathcal{A}_N -comodule is projective. Hence $\hat{\pi}$ has a comodule splitting $s : \mathcal{A}_N \rightarrow \mathcal{A}_G$ with $\hat{\pi}s = id_{\mathcal{A}_N}$.

Let $x \in \mathcal{A}_{G/N}^+$. It is straightforward to verify that $\pi|_{\mathcal{A}_{G/N}}$ is the counit of $\mathcal{A}_{G/N}$ (cf. remark (a) following Definition 2.2 in [25]). Hence $\hat{\pi}(x) = 0$, and therefore $\mathcal{A}_{G/N}^+ \mathcal{A}_G \subset \ker(\hat{\pi})$. It remains to show that $\ker(\hat{\pi}) \subset \mathcal{A}_{G/N}^+ \mathcal{A}_G$.

Define a linear map ϕ on \mathcal{A}_G by $\phi = (s\hat{\pi}) * S = m(s\hat{\pi} \otimes S)\Delta_G$, where m and S are respectively the multiplication map and antipodal map of \mathcal{A}_G . We show that $\phi(\mathcal{A}_G) \subset \mathcal{A}_{G/N}$. To see this, let $a \in \mathcal{A}_G$. Using the fact that s is a comodule morphism, i.e. $\alpha s = (s \otimes id)\beta$ or $(id \otimes \hat{\pi})\Delta_G s = (s \otimes id)\Delta_N$, along with properties of Hopf algebras morphisms, we obtain

$$\begin{aligned}
 (id \otimes \pi)\Delta_G(\phi(a)) &= (id \otimes \pi)\Delta_G(\sum s\pi(a_{(1)})S(a_{(2)})) \\
 &= (id \otimes \pi)(\sum \Delta_G(s\pi(a_{(1)}))\Delta_G(S(a_{(2)}))) \\
 &= (id \otimes \pi)(\sum \Delta_G(s\pi(a_{(1)}))(S(a_{(3)}) \otimes S(a_{(2)}))) \\
 &= \sum [(id \otimes \pi)(\Delta_G(s\pi(a_{(1)})))](S(a_{(3)}) \otimes \pi S(a_{(2)})) \\
 &= \sum (s \otimes id)(\Delta_N(\pi(a_{(1)})))(S(a_{(3)}) \otimes \pi S(a_{(2)})) \\
 &= \sum [s\pi(a_{(1)}) \otimes \pi(a_{(2)})][S(a_{(4)}) \otimes \pi S(a_{(3)})] \\
 &= \sum s\pi(a_{(1)})S(a_{(4)}) \otimes \pi(a_{(2)})S(a_{(3)}) \\
 &= \sum s\pi(a_{(1)})S(a_{(3)}) \otimes \varepsilon(a_{(2)}) \\
 &= \sum s\pi(a_{(1)})S(\varepsilon(a_{(2)})a_{(3)}) \otimes 1 \\
 &= \phi(a) \otimes 1,
 \end{aligned}$$

which means that $\phi(\mathcal{A}_G) \subset \mathcal{A}_{G/N}$, where ε is the co-unit on \mathcal{A}_G .

Next we observe that

$$s\hat{\pi} = (s\hat{\pi}) * \varepsilon = (s\hat{\pi}) * (S * id) = ((s\hat{\pi}) * S) * id = \phi * id,$$

and

$$\begin{aligned}
 \varepsilon\phi(a) &= \varepsilon(\sum s\pi(a_{(1)})S(a_{(2)})) = \sum \varepsilon(s\pi(a_{(1)}))\varepsilon(S(a_{(2)})) \\
 &= \sum \varepsilon[s\pi(a_{(1)})\varepsilon(a_{(2)})] = \varepsilon(s\pi(a)) = (\varepsilon_N\pi)(s\pi(a)) \\
 &= \varepsilon_N(\pi(a)) = \varepsilon(a),
 \end{aligned}$$

where $\hat{\pi}s = id_{\mathcal{A}_N}$ is used. The above means $\varepsilon\phi = \varepsilon$. From these we obtain

$$\begin{aligned} id - s\hat{\pi} &= \varepsilon * id - \phi * id = (\varepsilon - \phi) * id \\ &= (\varepsilon\phi - \phi) * id = [(\varepsilon - id)\phi] * id. \end{aligned}$$

Furthermore $(id - s\hat{\pi})(a) = a$ if $a \in \ker(\hat{\pi})$, we have $\ker(\hat{\pi}) \subset \text{Im}(id - s\hat{\pi})$. Therefore to show $\ker(\hat{\pi}) \subset \mathcal{A}_{G/N}^+ \mathcal{A}_G$, it suffices to show that $\text{Im}(id - s\hat{\pi}) \subset \mathcal{A}_{G/N}^+ \mathcal{A}_G$. Since $(\varepsilon - id)\phi(\mathcal{A}_G) \subset \mathcal{A}_{G/N}^+$ (because $\phi(\mathcal{A}_G) \subset \mathcal{A}_{G/N}$), the later follows from the identity

$$id - s\hat{\pi} = [(\varepsilon - id)\phi] * id = m((\varepsilon - id)\phi \otimes id)\Delta_G.$$

This proves Lemma 3.3. \square

Remarks. Using the notation Φ and Ψ of Schneider [14], Lemma 3.3 above can be restated as saying that the map Φ is the left inverse of Ψ , here $\Psi(\ker(\hat{\pi})) := \mathcal{A}_{G/N}$ and $\Phi(\mathcal{A}_{G/N}) := \mathcal{A}_G \mathcal{A}_{G/N}^+$. The first result of this kind is due to Takeuchi [16] for *commutative* Hopf algebras in which the fundamental theorem of affine algebraic group schemes [8] is given a purely algebraic proof.

In the language of Andruskiewitsch *et al* [1], Lemma 3.3 implies that the sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1,$$

or the sequence

$$\mathbb{C} \longrightarrow \mathcal{A}_{G/N} \longrightarrow \mathcal{A}_G \longrightarrow \mathcal{A}_N \longrightarrow \mathbb{C},$$

is exact. Note that in the purely algebraic situation of Parshall and Wang [11], for a given normal quantum subgroup in their sense (i.e. a-normal as defined in our paper here), the existence of an exact sequence of is not known and the uniqueness does not hold in general (cf. 1.6 and 6.3 loc. cit.). Lemma 3.3 above shows that such complication do not present themselves in the world of compact quantum groups: when we have a normal quantum group, we always have a unique exact sequence. This property might help to formulate an appropriate notion of quantum groups in algebraic setting.

Note also that the notion of exact sequence of quantum groups in [14] is equivalent to that in [1] under certain faithful (co)flat conditions. Note that an arbitrary Hopf algebra needs not to be faithfully flat over its Hopf subalgebras, according to a counter example of Schauenburg [13] constructed in response to Question 3.5.4 of Montgomery [10]. Recently, Chirvasitu [4] announced that the dense Hopf algebra \mathcal{A}_G associated with a compact

quantum group G is faithfully flat over its Hopf subalgebras, as the author had conjectured in [25] (cf. first part of Conjecture 1 on p3329 there). Chirvasitu's result can be used with those of Schneider [14] to give conclude that the map Φ is also the right inverse of Ψ , complementing Lemma 3.3.

We now prove the first main result of this paper.

Proof of Theorem 2.4 (cf. Schneider [14]). If N is normal, we show that N is a-normal.

Let $a \in \ker(\hat{\pi})$. We show that $ad_l(a) \in \ker(\hat{\pi}) \otimes \mathcal{A}_G$, where

$$ad_l(a) = \sum a_{(2)} \otimes a_{(1)} S(a_{(3)}).$$

By Lemma 3.3

$$\ker(\hat{\pi}) = \mathcal{A}_{G/N}^+ \mathcal{A}_G = \mathcal{A}_G \mathcal{A}_{G/N}^+ = \mathcal{A}_G \mathcal{A}_{G/N}^+ \mathcal{A}_G.$$

We assume without loss of generality $a = bc$ for $b \in \mathcal{A}_G$ and $c \in \mathcal{A}_{G/N}^+$. Then modulo $\ker(\hat{\pi}) \otimes \mathcal{A}_G$ we have

$$\begin{aligned} ad_l(a) &= \sum a_{(2)} \otimes a_{(1)} S(a_{(3)}) = \sum b_{(2)} c_{(2)} \otimes b_{(1)} c_{(1)} S(c_{(3)}) S(b_{(3)}) \\ &= \sum b_{(2)} \varepsilon(c_{(2)}) \otimes b_{(1)} c_{(1)} S(c_{(3)}) S(b_{(3)}) \\ &= \sum b_{(2)} \otimes b_{(1)} \sum (c_{(1)} S(c_{(2)})) S(b_{(3)}) \\ &= \sum b_{(2)} \otimes b_{(1)} \varepsilon(c) S(b_{(3)}) = 0, \end{aligned}$$

i.e., $ad_l(a) \in \ker(\hat{\pi}) \otimes \mathcal{A}_G$, where the property

$$(\varepsilon - id)(\mathcal{A}_{G/N}) \subset \mathcal{A}_{G/N}^+ \subset \ker(\hat{\pi})$$

is used, as well as the counital and antipodal properties.

Similarly, $ad_r(a) \in \ker(\hat{\pi}) \otimes \mathcal{A}_G$ by assuming $a = cb$ for $b \in \mathcal{A}_G$ and $c \in \mathcal{A}_{G/N}^+$, where

$$ad_r(a) = \sum a_{(2)} \otimes S(a_{(1)}) a_{(3)}.$$

Hence $ad_l(\ker(\hat{\pi})) \subset \ker(\hat{\pi}) \otimes \mathcal{A}_G$ and $ad_r(\ker(\hat{\pi})) \in \ker(\hat{\pi}) \otimes \mathcal{A}_G$, and N is a-normal.

Conversely, assume N is a-normal. We show that N is normal. Let $a \in \mathcal{A}_{N \setminus G}$, i.e.,

$$\sum a_{(1)} \otimes a_{(2)} - 1 \otimes a \in \ker(\hat{\pi}) \otimes \mathcal{A}_G.$$

Applying $(ad_l \otimes id)$ to this, we obtain

$$\sum a_{(2)} \otimes a_{(1)} S(a_{(3)}) \otimes a_{(4)} - 1 \otimes 1 \otimes a \in \ker(\hat{\pi}) \otimes \mathcal{A}_G \otimes \mathcal{A}_G.$$

Multiplying the second and the third factors, we obtain

$$\sum a_{(2)} \otimes a_{(1)} S(a_{(3)}) a_{(4)} - 1 \otimes a \in \ker(\hat{\pi}) \otimes \mathcal{A}_G.$$

By the antipodal and counital properties,

$$\sum a_{(2)} \otimes a_{(1)} S(a_{(3)}) a_{(4)} = \sum a_{(2)} \otimes a_{(1)} \varepsilon(a_{(3)}) = \sum a_{(2)} \otimes a_{(1)}.$$

Hence

$$\sum a_{(2)} \otimes a_{(1)} - 1 \otimes a \in \ker(\hat{\pi}) \otimes \mathcal{A}_G, \text{ and therefore}$$

$$\sum a_{(1)} \otimes a_{(2)} - a \otimes 1 \in \mathcal{A}_G \otimes \ker(\hat{\pi}), \text{ i.e., } a \in \mathcal{A}_{G/N}.$$

Similarly, if $a \in \mathcal{A}_{G/N}$, then $a \in \mathcal{A}_{N \setminus G}$.

Hence $\mathcal{A}_{G/N} = \mathcal{A}_{N \setminus G}$. By Proposition 3.2.(3)', N is normal. \square

Since the notions of conormal in [14] and a-normal are the same concepts, Theorem 2.4 enables it to apply related results in [14] to our normal quantum subgroups.

4. THIRD FUNDAMENTAL ISOMORPHISM THEOREM FOR QUANTUM GROUPS AND PROPERTIES F AND FD

In this section, we give several applications of Theorem 2.4. Because of this and other results in sections 2 and 3, we focus on the dense Hopf $*$ -subalgebras associated to compact quantum groups in this section.

The three fundamental isomorphism theorems in the theory of groups are foundational results on structure of groups. One way naturally expect their analogs to be valid in the theory of quantum groups. Unfortunately, quantum analog of the first fundamental isomorphism theorem is not always true for epimorphism of quantum groups (i.e. injection of Hopf algebras) except for the situation where exact sequence can be constructed, cf. [11, 14, 17, 1]. For instance, not every Woronowicz C^* -subalgebra of A_G is of the form $A_{G/N}$ with N a normal quantum subgroup of G . This already fails when A_G is the group C^* -algebra $C^*(F_2)$ of the free group F_2 on two generators, as a Woronowicz C^* -subalgebra of A_G is of the form $A_{G/N}$ if and only if it is the group C^* -algebra $C^*(\Gamma)$ of a normal subgroup Γ of F_2 . It is not clear how a quantum analog of the second fundamental isomorphism theorem can be formulated.

However, on the bright side, as an application of Theorem 2.4, we have the following complete analog of the *third fundamental isomorphism theorem* for compact quantum groups.

Theorem 4.1. *Let (N, π) be a normal quantum subgroup of G . Let (H, θ) be a quantum subgroup of G that contains (N, π) , i.e., there is a morphism π_1 from A_H to A_N such that (N, π_1) is a quantum subgroup of H with $\pi = \pi_1\theta$. Then (N, π_1) is normal in H . If furthermore H is normal in G and let $\theta' = \theta|_{A_{G/N}}$, the restriction of θ to $A_{G/N}$, then $(H/N, \theta')$ is normal in G/N and*

$$A_{(G/N)/(H/N)} = A_{G/H}$$

Proof. Let $z \in \mathcal{A}_H$ be such that $\pi_1(z) = 0$. Assume $z = \theta(x)$ for some $x \in \mathcal{A}_G$. Then $\pi(x) = 0$. Since N is a-normal in G by Theorem 2.4, we have

$$\sum \pi(x_{(2)}) \otimes S(x_{(1)})x_{(3)} = 0, \quad \sum \pi(x_{(2)}) \otimes x_{(1)}S(x_{(3)}) = 0.$$

Hence

$$\sum \pi_1\theta(x_{(2)}) \otimes \theta(S(x_{(1)})x_{(3)}) = 0, \quad \sum \pi_1\theta(x_{(2)}) \otimes \theta(x_{(1)}S(x_{(3)})) = 0.$$

That is

$$\sum \pi_1(z_{(2)}) \otimes S(z_{(1)})z_{(3)} = 0, \quad \sum \pi_1(z_{(2)}) \otimes z_{(1)}S(z_{(3)}) = 0.$$

This means that (π_1, N) is a-normal and is therefore normal in H by Theorem 2.4.

Let $a \in \mathcal{A}_{G/N}$. Then it is immediate to verify that $\theta(a) \in \mathcal{A}_{H/N}$, and therefore $\theta(\mathcal{A}_{G/N})$ is contained in $\mathcal{A}_{H/N}$. Conversely, let $b \in \mathcal{A}_{H/N}$. Assume $b = \theta(a)$ for some $a \in \mathcal{A}_G$. Put

$$\bar{a} = E_{G/N}(a) = (id \otimes h_N\pi)\Delta_G(a).$$

Then $\bar{a} \in \mathcal{A}_{G/N}$ and

$$b = E_{H/N}(b) = (id \otimes h_N\pi_1)\Delta_H(b)$$

by the remarks before Proposition 3.1 applied to G/N and H/N respectively. Moreover, we have

$$\begin{aligned} \theta(\bar{a}) &= (\theta \otimes h_N\pi)\Delta_G(a) = (\theta \otimes h_N\pi_1\theta)\Delta_G(a) \\ &= (id \otimes h_N\pi)\Delta_H(\theta(a)) = (id \otimes h_N\pi_1)\Delta_H(b) \\ &= b. \end{aligned}$$

That is $\theta(\bar{a}) = b$. Therefore $\theta(\mathcal{A}_{G/N}) = \mathcal{A}_{H/N}$.

Now assume (H, θ) is normal. For ease of notation, let $G' = G/N$ and $H' = H/N$. The above shows that (H', θ') is a quantum subgroup of G' . If $a \in \mathcal{A}_{G'/H'}$, that is, $(id \otimes \theta')\Delta(a) = a \otimes 1_{H'}$, then it is clear that a is in $\mathcal{A}_{G/H}$ since $\theta' = \theta|_{A_{G/N}}$ and $1_{H'} = 1_H$. Conversely, if $a \in \mathcal{A}_{G/H}$, that is

$(id \otimes \theta)\Delta(a) = a \otimes 1_H$, then $(id \otimes \pi_1\theta)\Delta(a) = a \otimes 1_N$. This means that $a \in \mathcal{A}_{G'} = \mathcal{A}_{G/N}$. But then we have

$$(id \otimes \theta')\Delta(a) = (id \otimes \theta)\Delta(a) = a \otimes 1_H = a \otimes 1_{H'}.$$

Hence $a \in \mathcal{A}_{G'/H'}$ and $\mathcal{A}_{G'/H'} = \mathcal{A}_{G/H}$.

The result is completely proved. \square

Remark: Instead of an isomorphism such as in $(G/N)/(H/N) \cong G/H$ in group theory, we have *an exact equality* of quantum function algebras in Theorem 4.1 above.

The proof of Theorem 4.1 actually yields the following stronger result without assuming N to be normal:

Theorem 4.1' *Let (N, π) be a (not necessarily normal) quantum subgroup of G . Let (H, θ) be a quantum subgroup of G that contains (N, π) , i.e., there is a morphism π_1 such that (N, π_1) is a quantum subgroup of H with $\pi = \pi_1\theta$. Then*

$$\mathcal{A}_{(G/N)/(H/N)} = \mathcal{A}_{G/H}, \quad \text{and} \quad \mathcal{A}_{(N \setminus H) \setminus (N \setminus G)} = \mathcal{A}_{H \setminus G}.$$

For other applications of Theorem 2.4, we first recall the following properties of compact quantum groups (cf. [25]).

Definition 4.2. *A compact quantum group G is said to have **property F** if each Woronowicz C^* -subalgebra of \mathcal{A}_G is of the form $\mathcal{A}_{G/N}$ for some normal quantum subgroup N of G ; it is said to have **property FD** if each quantum subgroup of G is normal.*

*A compact quantum group is called **almost classical** if its representation ring $R(G)$ is order isomorphic to the representation ring of a compact group.*

We note that the notions property F and property FD above can be defined almost verbatim for all Hopf algebras—one only need to replace the words “compact quantum group” (resp. “Woronowicz C^* -subalgebra”) with the words “Hopf algebra” (resp. “Hopf subalgebra”) in the above definition. These notions are motivated by the following facts (see Propositions 2.3 and 2.4 in [25]): if G is a compact group, then its function algebra $C(G) = \mathcal{A}_G$ has property F ; if G is the dual of a discrete quantum group Γ , then its quantum function algebra $C^*(\Gamma)$ has property FD . Therefore compact quantum groups with property F and almost classical compact quantum groups are closest to compact groups, while compact quantum groups with

property FD are closest to the compact quantum group dual of discrete groups. All the quantum groups obtained by deformation of compact Lie groups and all universal quantum groups (except $A_u(Q)$) are almost classical and possess property F (see [25]). Furthermore, we have following result on the structure of compact quantum group with property F .

Theorem 4.3. *Let G be a compact quantum group with property F . Then*

- (1) *its quantum subgroup (H, θ) also has property F ;*
- (2) *quotient group G/N by a normal quantum subgroup (N, π) also has property F .*

Proof. (1) Let $\mathcal{B} \subset \mathcal{A}_H$ be a Hopf subalgebra. Let $\mathcal{B}' = \theta^{-1}(\mathcal{B})$ be the inverse image of \mathcal{B} in \mathcal{A}_G under θ , which is a Hopf $*$ -subalgebra of the latter. By property F of G , assume that $\mathcal{B}' = \mathcal{A}_{G/N'}$ for some normal quantum subgroup (N', π') of G . Put $\mathcal{I}' = \ker(\hat{\pi}')$ and $\mathcal{I} = \theta(\mathcal{I}')$.

We show that \mathcal{I} is a Hopf ideal in \mathcal{A}_H and that by Theorem 2.4

$$\mathcal{A}_N := \mathcal{A}_H / \mathcal{I}$$

defines a normal quantum subgroup (N, π) of H such that $\mathcal{B} = \mathcal{A}_{H/N}$, where π is the natural surjection from \mathcal{A}_H to \mathcal{A}_N . This will prove part (1) of the theorem.

Since θ preserves the $*$ -operations, \mathcal{I} is a $*$ -ideal. Let $x = \theta(x') \in \mathcal{I}$, where $x' \in \mathcal{I}'$. Then

$$\begin{aligned} \Delta_H(x) &= (\theta \otimes \theta) \Delta_G(x') \in (\theta \otimes \theta)(\mathcal{A}_G \otimes \mathcal{I}' + \mathcal{I}' \otimes \mathcal{A}_G) \\ &= \mathcal{A}_H \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{A}_H. \end{aligned}$$

One also easily verifies that

$$S_H(x) = S_H(\theta(x')) = \theta(S_G(x')) \in \mathcal{I}, \quad \text{and}$$

$$\varepsilon_H(x) = \varepsilon_H(\theta(x')) = \varepsilon_G(x') = 0.$$

Hence \mathcal{I} is a Hopf $*$ -ideal of \mathcal{A}_H .

Let $b = \theta(b') \in \mathcal{B}$, where $b' \in \mathcal{B}' = \mathcal{A}_{G/N'}$. Then,

$$(id \otimes \pi') \Delta_G(b') = b' \otimes 1_{N'}, \quad \text{i.e.,} \quad \Delta_G(b') - b' \otimes 1 \in \mathcal{A}_G \otimes \mathcal{I}'.$$

Applying $(\theta \otimes \theta)$ to the latter, we obtain

$$\Delta_H(b) - b \otimes 1_H \in \mathcal{A}_H \otimes \mathcal{I},$$

i.e., $\mathcal{B} \subset \mathcal{A}_{H/N}$. Conversely, let $b = \theta(b') \in \mathcal{A}_{H/N}$, where $b' \in \mathcal{A}_G$. Then

$$\Delta_H(b) - b \otimes 1_H \in \mathcal{A}_H \otimes \mathcal{I}, \quad \text{i.e.,} \quad (\theta \otimes \theta)(\Delta_G(b') - b' \otimes 1) \in \mathcal{A}_H \otimes \mathcal{I}.$$

Hence

$$(\Delta_G(b') - b' \otimes 1) \in (\theta \otimes \theta)^{-1}(\mathcal{A}_H \otimes \mathcal{I}) = \mathcal{A}_G \otimes \mathcal{I}',$$

i.e., $b' \in \mathcal{B}'$ and therefore $b = \pi'(b') \in \mathcal{B}$. This shows that $\mathcal{B} = \mathcal{A}_{H/N}$, and hence that N is normal by Proposition 3.2. Proof of part (1) is finished.

(2) Let (N, π) be a normal quantum subgroup of G and let $\mathcal{C} \subset \mathcal{A}_{G/N}$ be a Hopf subalgebra of $\mathcal{A}_{G/N}$ and therefore also a Hopf subalgebra of \mathcal{A}_G . We show that there is a normal quantum subgroup (K, π_0) of G/N such that $\mathcal{C} = \mathcal{A}_{(G/N)/K}$.

By property F of G , let (N_1, π_1) be a normal quantum subgroup of G such that $\mathcal{C} = \mathcal{A}_{G/N_1}$. Let $\mathcal{I}_1 = \ker(\hat{\pi}_1)$, where $\hat{\pi}_1$ is the morphism from \mathcal{A}_G to \mathcal{A}_{N_1} associated with π_1 . Put $\mathcal{I}_0 := \mathcal{I}_1 \cap \mathcal{A}_{G/N}$.

We first verify that \mathcal{I}_0 is a Hopf $*$ -ideal in $\mathcal{A}_{G/N}$. Let $x \in \mathcal{I}_0$. Since $\mathcal{A}_{G/N}$ is a Hopf algebra, $\Delta(x) \in \mathcal{A}_{G/N} \otimes \mathcal{A}_{G/N}$. Hence

$$\begin{aligned} \Delta(x) &\in [\mathcal{I}_1 \otimes \mathcal{A}_G + \mathcal{A}_G \otimes \mathcal{I}_1] \cap [\mathcal{A}_{G/N} \otimes \mathcal{A}_{G/N}] \\ &= \mathcal{I}_0 \otimes \mathcal{A}_{G/N} + \mathcal{A}_{G/N} \otimes \mathcal{I}_0. \end{aligned}$$

Similarly, one verifies $S(x) \in \mathcal{I}_0$ and $\varepsilon(x) = 0$. Hence \mathcal{I}_0 is a Hopf $*$ -ideal in $\mathcal{A}_{G/N}$.

Furthermore, by Theorem 2.2, Proposition 3.2 and Theorem 2.4, $\mathcal{I}_1 = \ker(\hat{\pi}_1)$ is a normal Hopf ideal in \mathcal{A}_G in the sense of Definition 2.3. Hence

$$ad_l(x) \in [\mathcal{I}_1 \otimes \mathcal{A}_G] \cap [\mathcal{A}_{G/N} \otimes \mathcal{A}_{G/N}] = \mathcal{I}_0 \otimes \mathcal{A}_{G/N}.$$

Similarly $ad_r(x) \in \mathcal{I}_0 \otimes \mathcal{A}_{G/N}$. This shows that \mathcal{I}_0 is a normal Hopf ideal in $\mathcal{A}_{G/N}$.

By Theorem 2.4, there exists a normal quantum subgroup (K, π_0) of G/N such that $\mathcal{A}_K = (\mathcal{A}_{G/N})/\mathcal{I}_0$, where $\ker(\pi_0) = \overline{\mathcal{I}_0}$ (cf. Theorem 2.2).

We show that $\mathcal{C} = \mathcal{A}_{(G/N)/K}$. Let $c \in \mathcal{C} (= \mathcal{A}_{G/N_1}) \subset \mathcal{A}_{G/N}$. Then

$$\Delta(c) - c \otimes 1 \in \mathcal{A}_G \otimes \mathcal{I}_1, \quad \text{and} \quad \Delta(c) \in \mathcal{A}_{G/N} \otimes \mathcal{A}_{G/N}.$$

Hence

$$\Delta(c) - c \otimes 1 \in (\mathcal{A}_{G/N} \otimes \mathcal{A}_{G/N}) \cap (\mathcal{A}_{G/N} \otimes \mathcal{I}_1) = \mathcal{A}_{G/N} \otimes \mathcal{I}_0.$$

That is $\mathcal{C} \subset \mathcal{A}_{(G/N)/K}$. Conversely, let $c \in \mathcal{A}_{(G/N)/K}$. Then arguing backwards, we obviously have $c \in \mathcal{C} (= \mathcal{A}_{G/N_1})$. This shows that $\mathcal{C} = \mathcal{A}_{(G/N)/K}$, proving part (2). \square

Remarks. (1) With notation in the first part of the proof of Theorem 4.3, the

following gives a well defined morphism θ_0 of Woronowicz Hopf C^* -algebras from $A_{N'}$ to A_N that makes the relevant diagram commutative:

$$\theta_0(\pi'(a)) := \pi(\theta(a)), \quad a \in A_G.$$

(2) We note that though there are more Woronowicz C^* -ideals in A_G than Hopf $*$ -ideals in \mathcal{A}_G in the correspondence of Theorem 2.2 (see Remark (a) after Lemma 4.3 in [25]), Woronowicz C^* -subalgebras of A_G and Hopf subalgebras of \mathcal{A}_G are in bijective correspondence: every Woronowicz C^* -subalgebra B of A_G uniquely corresponds to its canonical dense Hopf $*$ -subalgebra \mathcal{B} of B .

No definitive answer seems to be known to the following question yet:

Question 4.4. *Is a Woronowicz C^* -subalgebras of a full Woronowicz C^* -algebra necessarily full?*

A related question is the following one on the relation between a Woronowicz ideal of a Woronowicz C^* -subalgebra and the ideal it generates in the original Woronowicz C^* -algebra. We formulate two versions of this. Assuming A is a Woronowicz C^* -algebra and A_0 a Woronowicz C^* -subalgebra of A . As usual, \mathcal{A} and \mathcal{A}_0 denote the Hopf $*$ -algebras of A and A_0 respectively.

Question 4.5. (a) *Let I_0 be a Woronowicz C^* -ideal of A_0 . Let $I := AI_0A$ be the Woronowicz C^* -ideal of A generated by I_0 . Is the identity $I_0 = I \cap A_0$ always true?*

(b) *Let \mathcal{I}_0 be a Hopf $*$ -ideal of \mathcal{A}_0 . Let $\mathcal{I} := \mathcal{A}\mathcal{I}_0\mathcal{A}$ be the Hopf $*$ -ideal of \mathcal{A} generated by \mathcal{I}_0 . Is the identity $\mathcal{I}_0 = \mathcal{I} \cap \mathcal{A}_0$ always true?*

In the case of a function algebra $A = C(G)$ on a compact group G , A_0 is the algebra of functions on a quotient group G/N , I_0 defines a subgroup H/N of the quotient group, and I corresponds to the pull back H of H/N in G . Because of Theorem 2.2, parts (a) and (b) of Question 4.5 are equivalent.

We note that an affirmative answer to this question corresponds the opposite direction to Lemma 3.3 in the following sense: In Lemma 3.3, the big ideal $\hat{\pi}$ is recovered from the smaller ideal $\mathcal{A}_{G/N}^+$ whereas in Question 4.5, the small ideal \mathcal{I}_0 is recovered from the big ideal \mathcal{I} . It would be of interest to understand the more precise relationship.

Note also that part (a) of Question 4.5 can be formulated for arbitrary C^* -algebras. Similarly, part (b) can be formulated for more general rings and algebras, though the corresponding answer is very likely negative.

Definition 4.6. *We say a compact quantum group G has **pull back property** if the answer for Question 4.5 is affirmative for $A = A_G$, equivalently for $\mathcal{A} = \mathcal{A}_G$.*

Just as every subgroup of G/N is of the form H/N for some subgroup H of G containing N , it our believe that all compact quantum groups have pull back property, though this may not be true for more general Hopf algebras.

We are ready to prove the following result on property FD .

Theorem 4.7. *Let G be a compact quantum group with property FD . Then*

- (1) *quantum subgroup (H, θ) of G also has property FD ;*
- (2) *quotient group G/N by a normal quantum subgroup (N, π) also has property FD provided G has pull back property.*

Proof. (1) Let (H_1, π_1) be a quantum subgroup of H . Then $(H_1, \pi_1\theta)$ is a quantum subgroup of G that is contained in (H, θ) . By property FD , $(H_1, \pi_1\theta)$ is normal in G . By Theorem 4.1, (H_1, π_1) is normal in H . This shows that H has property FD .

(2) Assume G is a compact quantum group with the pull back property in addition to property FD .

Let (N, π) be a normal quantum subgroup of G and let (K, π_0) be a quantum subgroup of G/N . Let \mathcal{I}_0 be the kernel of $\hat{\pi}_0$ in $\mathcal{A}_{G/N}$ and \mathcal{I}_N the kernel of $\hat{\pi}$ in \mathcal{A}_G . Identify \mathcal{A}_K with $(\mathcal{A}_{G/N})/\mathcal{I}_0$. Put $\mathcal{I} := \mathcal{A}\mathcal{I}_0\mathcal{A}$. Then \mathcal{I} is a Hopf $*$ -ideal in \mathcal{A}_G and defines a quantum subgroup (H, θ) of G , where

$$\hat{\theta} : \mathcal{A}_G \rightarrow \mathcal{A}_G/\mathcal{I}, \quad \mathcal{A}_H := \mathcal{A}_G/\mathcal{I}.$$

By the co-unital property $\varepsilon_{G/N} = \varepsilon_K\pi_0$ and the fact that $\varepsilon_{G/N}$ is the restriction of π to $\mathcal{A}_{G/N}$, we have $\pi = \varepsilon_K\pi_0$ on $\mathcal{A}_{G/N}$. It follows that $\mathcal{I}_0 \subset \mathcal{I}_N$ and therefore we have an inclusion $\mathcal{I} \subset \mathcal{I}_N$. This means that H is a quantum subgroup of G containing N and (N, π_1) is normal in H as shown in Theorem 4.1, where π_1 is defined by $\pi_1(\theta(a)) := \pi(a)$, for $a \in \mathcal{A}_G$. On the other hand, every element in \mathcal{A}_K is of the form $\pi_0(a)$ for some $a \in \mathcal{A}_{G/N}$. If $\pi_0(a) = 0$, then the above inclusion implies $\theta(a) = 0$. Hence

$$\rho(\pi_0(a)) := \theta(a)$$

gives a well defined morphism from \mathcal{A}_K to \mathcal{A}_H , where $\pi_0(a) \in \mathcal{A}_K$. It can be checked that ρ is a morphism of Hopf algebras.

We summarize all the morphisms in the following commutative diagram.

$$\begin{array}{ccccccc}
 \mathbb{C} & \longrightarrow & \mathcal{A}_{G/N} & \longrightarrow & \mathcal{A}_G & \xrightarrow{\hat{\pi}} & \mathcal{A}_N \xrightarrow{\varepsilon_N} \mathbb{C} \\
 & & \downarrow \hat{\pi}_0 & & \downarrow \hat{\theta} & & \parallel \\
 \mathbb{C} & \longrightarrow & \mathcal{A}_K & \xrightarrow{\hat{\rho}} & \mathcal{A}_H & \xrightarrow{\hat{\pi}_1} & \mathcal{A}_N \xrightarrow{\varepsilon_N} \mathbb{C}
 \end{array}$$

Moreover, the image of ρ is in $\mathcal{A}_{H/N}$ because by property of Hopf algebra morphisms, for $a \in \mathcal{A}_{G/N}$ we have

$$\begin{aligned}
 (id_H \otimes \pi_1)\Delta_H(\rho(\pi_0(a))) &= (id_H \otimes \pi_1)\Delta_H(\theta(a)) \\
 &= (\theta \otimes \pi_1)\Delta_G(a) = (\theta \otimes id_N)(id_G \otimes \pi)\Delta_G(a) \\
 &= (\theta \otimes id_N)(a \otimes 1_N) = \theta(a) \otimes 1_N = \rho(\pi_0(a)) \otimes 1_N,
 \end{aligned}$$

which means $\rho(\pi_0(a)) \in \mathcal{A}_{H/N}$.

In addition, if $\theta(a) = 0$ for some a in $\mathcal{A}_{G/N}$, then a is $\mathcal{I} \cap \mathcal{A}_{G/N}$, which is \mathcal{I}_0 by assumption, and $\pi_0(a) = 0$. Hence $\ker(\rho) = 0$ and ρ is an injection. As in the proof of Theorem 4.1, $\theta(\mathcal{A}_{G/N}) = \mathcal{A}_{H/N}$. Therefore, ρ is also a surjection. Hence we have an isomorphism $\rho(\mathcal{A}_K) = \mathcal{A}_{H/N}$.

Since H is normal in G by property FD of G , Theorem 4.1 guarantees that K (which is H/N) is normal in G/N and the proof is complete. \square

5. OTHER PROPERTIES OF NORMAL QUANTUM SUBGROUPS

Just as in the case of ordinary groups, we have

Proposition 5.1. *Let G_1, G_2 be compact quantum groups. Let π_1 be the natural embedding of G_1 into $G_1 \times G_2$ defined by the surjection*

$$\pi_1 : A_{G_1} \otimes A_{G_2} \rightarrow A_{G_1},$$

$$\pi_1 = id_1 \otimes \varepsilon_2, \quad \varepsilon_2 \text{ being the counit of } A_{G_2}.$$

Then (G_1, π_1) is a normal quantum subgroup of $G = G_1 \times G_2$ and $G/G_1 \cong G_2$.

Proof. According to §2 and using the properties of the counit and Haar measure, one obtains immediately

$$A_{G/G_1} = E_{G/G_1}(A_{G_1} \otimes A_{G_2}) = 1 \otimes A_{G_2}$$

which is a Woronowicz C^* -subalgebra isomorphic to A_{G_2} . Hence the proposition follows from part (2) of Proposition 3.2. \square

For free product [20], the situation is quite different from tensor product:

Proposition 5.2. *Let G_1, G_2 be compact quantum groups (not necessarily duals of discrete groups). Let $G = G_1 \hat{*} G_2$ be the free product compact quantum group underlying $A_{G_1} * A_{G_2}$. Let π_1 be the natural embedding of G_1 into G defined by the surjection*

$$\pi_1 : A_{G_1} * A_{G_2} \rightarrow A_{G_1},$$

$$\pi_1 = id_1 * \varepsilon_2.$$

*If G_1 has at least one irreducible representation of dimension greater than one, then (G_1, π_1) is **not** a normal quantum subgroup of $G_1 \hat{*} G_2$. Otherwise, (G_1, π_1) is normal in $G_1 \hat{*} G_2$.*

The hat in the symbol $\hat{*}$ above signifies the “Fourier transform” of $*$.

Proof. Let

$$u = \sum_{ij} e_{ij}^u \otimes u_{ij}, \quad v = \sum_{kl} e_{kl}^v \otimes v_{kl}$$

be irreducible representations of G_1, G_2 respectively. Assume that the dimension of u is greater than one. Let $\bar{u} = \sum_{ij} e_{ij}^u \otimes u_{ij}^*$ denote the conjugate representation of u . Then by [20], the interior tensor product representation

$$u \otimes_{in} v \otimes_{in} \bar{u} = \sum_{ijklrs} e_{ij}^u \otimes e_{kl}^v \otimes e_{rs}^u \otimes u_{ij} v_{kl} u_{rs}^*$$

is irreducible. Let h_1 be the Haar state of G_1 . Then

$$h_1 \pi_1(u \otimes_{in} v \otimes_{in} \bar{u}) = \sum_{ijrs} e_{ij}^u \otimes I_v \otimes e_{rs}^u h_1(u_{ij} u_{rs}^*),$$

where I_v is the identity matrix acting on the Hilbert space of v . Since u is of dimension greater than one, $u \otimes \bar{u}$ properly contains the trivial representation of G_1 (with multiplicity one). Hence

$$h_1(u \otimes_{in} \bar{u}) = \sum_{ijrs} e_{ij}^u \otimes e_{rs}^u h_1(u_{ij} u_{rs}^*)$$

is neither the identity matrix, nor the zero matrix on $H_u \otimes H_{\bar{u}}$. (See also Theorem 5.7 of Woronowicz [26].) Therefore

$$h_1 \pi_1(u \otimes_{in} v \otimes_{in} \bar{u})$$

is neither the identity matrix, nor the zero matrix on $H_u \otimes H_v \otimes H_{\bar{u}}$. By Proposition 3.2.(4), G_1 is not normal.

If G_1 has no non-trivial irreducible representations of dimension greater than one, then by [26] G_1 is the compact quantum group dual of a discrete

group Γ , i.e., $A_{G_1} = C^*(\Gamma)$. By [20], every irreducible representation of G is of the form

$$w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n},$$

where w^{λ_i} is a non-trivial representation belonging to either the set Γ or the set \hat{G}_2 , w^{λ_i} and $w^{\lambda_{i+1}}$ being in different sets. It is clear that

$$\pi_1(w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n})$$

is a constant diagonal matrix, with the constant diagonal entry equal to the product of those w^{λ_i} 's that belong to Γ . Use $[w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n}]$ to denote this diagonal entry. Since $h_{G_1}(\gamma) = 0$ if γ is not the neutral element of Γ , One sees that Proposition 3.2.(4) is fulfilled. Hence (G_1, π_1) is normal in $G = G_1 \hat{*} G_2$ and A_{G/G_1} is equal to the closure of the linear span of entries of the matrix $w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n}$ such that $[w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n}]$ is the neutral element of Γ . \square

Similar to the second situation in the proposition above, we have the following

Proposition 5.3. *Consider a crossed product $A \rtimes_{\alpha} \Gamma$ of a compact quantum group A by a discrete group Γ [21]. Then $C^*(\Gamma)$ is a quantum normal subgroup of $A \rtimes_{\alpha} \Gamma$ with quotient A via the morphism*

$$\pi : A \rtimes_{\alpha} \Gamma \longrightarrow C^*(\Gamma), \quad \pi(a\gamma) = \varepsilon(a)\gamma,$$

where ε is the counit of A and $a \in A$, $\gamma \in \Gamma$.

More generally, one has

Proposition 5.4. *Let (A, Γ, α) be a Woronowicz C^* -dynamical system. Let $I \triangleleft A$ be an α -invariant Woronowicz C^* -ideal so that A/I is a normal quantum subgroup of A with quotient quantum group B . Let K be the kernel of the action α . Let $\tilde{\alpha}$ be the induced action of Γ/K on A/I . Then $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ is a normal quantum subgroup of $A \rtimes_{\alpha} \Gamma$ with quotient $B \rtimes_{\alpha} K$.*

Proof. Consider the morphisms

$$\pi : A \longrightarrow A/I \rtimes_{\tilde{\alpha}} \Gamma/K,$$

$$U : \Gamma \longrightarrow A/I \rtimes_{\tilde{\alpha}} \Gamma/K,$$

defined by $\pi(a) = \tilde{a}$, $U_{\gamma} = \tilde{\gamma}$. Here \tilde{a} is the image in A/I of an element $a \in A$ viewed as an element of $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ via inclusion; $\tilde{\gamma}$ is the image in

Γ/K of an element $\gamma \in \Gamma$ viewed as an element of $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ via inclusion. One can verify that (π, U) is a covariant representation, i.e.,

$$U_\gamma \pi(a) U_\gamma^{-1} = \pi(\alpha_\gamma(a)).$$

Hence there is a surjection

$$\pi \times U : A \rtimes_\alpha \Gamma \longrightarrow A/I \rtimes_{\tilde{\alpha}} \Gamma/K,$$

extending π and U . This morphism preserves the coproducts because its restrictions to A and $C^*(\Gamma)$ do. This shows that $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ is a quantum subgroup of $A \rtimes_\alpha \Gamma$ (true under only the assumption that A/I is a quantum subgroup of A).

Let $A = A_G$ and assume that $A/I = A_N$ is a normal quantum subgroup of A with quotient $B = A_{G/N}$. Let π_0 be the morphism from A_G to A_N . Let θ denote the surjection $\pi \times U$. Let $h = h_{A/I} \rtimes_{\tilde{\alpha}} h_{\Gamma/K}$ be the Haar state on $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ (cf. [21]). Then every irreducible representation of the quantum group $A \rtimes_\alpha \Gamma$ is of the form $(u_{ij}^\beta \gamma)$, where (u_{ij}^β) is an irreducible representation of dimension d_β of the quantum group A and $\gamma \in \Gamma$ (cf. [21] as well). Let $S(N)$ be the set of λ 's such that $h_N \pi_0(u^\lambda)$ is d_λ , the dimension of the irreducible representation u^λ (i.e. $S(N)$ is as in proof of (4) \Rightarrow (3) in of Proposition 2.1 in [25]). Then

$$(h\theta(u_{ij}^\beta \gamma)) = (h(\tilde{u}_{ij}^\beta)h(\tilde{\gamma})) = \begin{cases} I_{d_\beta} & \text{if } \beta \in S, \gamma \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Hence by Proposition 2.1 in [25] and its proof, we conclude that $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ is a normal quantum subgroup of $A \rtimes_\alpha \Gamma$ with quotient $B \rtimes_\alpha K$. \square

We note that A is not a normal quantum subgroup of $A \rtimes_\alpha \Gamma$, unlike the semi-direct product of groups.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GA 30602

FAX: 706-542-2573; TEL: 706-542-0884

E-mail address: `szwang@math.uga.edu`